# A TECHNIQUE FOR COMPUTING MINORS OF BINARY HADAMARD MATRICES AND APPLICATION TO THE GROWTH PROBLEM *

CHRISTOS KRAVVARITIS† AND MARILENA MITROULI†

**Abstract.** A technique for computing all possible minors of order $(n-j)\times(n-j)$ of binary Hadamard matrices with entries $(0,1)$ is introduced. The method exploits the property $S^TS = \frac{1}{4}(n+1)(I_n+J_n)$ of such matrices $S$, and also the symmetry and special block structure appearing if one forms $D^TD$, where $D$ is a submatrix of $S$. Theoretically, it works for every pair of values $n$ and $j$ and provides general analytical formulas. The whole process can be standardized and implemented as a computer algorithm. The usefulness of such a method is justified with the application to the growth problem. This study offers as a collateral result more insight into some structural properties of these matrices and leads to the formulation of the growth conjecture for binary Hadamard matrices.

**Key words.** Binary Hadamard matrices, determinant calculus, symbolic computations, Gaussian elimination, growth problem

**AMS subject classifications.** 15A15, 05B20, 65F40, 65F05, 65G50

## 1. Introduction.

**1.1. Orthogonal matrices and minors.** An *orthogonal matrix* $Q$ of order $n\times n$ satisfies $QQ^T = Q^TQ = I_n$. Due to this definition, orthogonal matrices have determinant $\pm 1$, the inverse of an orthogonal matrix is its transpose, the product of two orthogonal matrices is an orthogonal matrix and they yield the unitarily invariance property of the Euclidean matrix norm. Therefore it can be proved that they have some important numerical properties, e.g. the product of any matrix with an orthogonal matrix is always stable (i.e. gives only a small and acceptable error) and orthogonally similar matrices have the same eigenvalues.

In this paper we deal with generalized normalized orthogonal matrices, as described by Definition 1.1.

Definition 1.1. *A matrix $A = (a_{ij})$ is called* normalized *if* $\max_{i,j}|a_{ij}| = 1$. *We call a normalized $n\times n$ matrix $A$* normalized orthogonal *if $AA^T = A^TA = c(A)I_n$ and* generalized normalized orthogonal *if $AA^T = A^TA = c(A)(I_n+J_n)$ for some constant $c(A)$, where $J_n$ denotes the all ones matrix of order $n$.*

A similar definition was given in [4]. These matrices are generalized to within a row scaling, i.e. the product of such a matrix with its transpose gives a multiple either of the identity matrix $I_n$ (e.g. Hadamard and weighing matrices) or of the similarly special structured matrix $I_n + J_n$ (e.g. binary Hadamard matrices).

To be precise, we study the computation of minors of binary generalized normalized orthogonal matrices, having entries $(0,1)$. In general it is difficult to obtain *analytical formulas* for minors of various orders for a given arbitrary matrix. A very interesting result for computing *numerically* all principal minors of a matrix, yielding an $O(2^n)$ algorithm, was presented in [10]. For the orthogonal matrices under consideration in this work, the derivation of analytical formulas for their minors is possible due to their special structure and properties.

*This research was financially supported by ΠΕΝΕΔ 03ΕΔ 740 of the Greek General Secretariat for Research and Technology.

†Department of Mathematics, University of Athens, Panepistemiopolis, 15784 Athens, Greece (ckrav,mmitroul@math.uoa.gr).

TABLE 1.1
*Values of minors for Hadamard matrices of general order n*

| minor | values of minors |
|---|---|
| $n-1$ | $n^{n/2-1}$ |
| $n-2$ | 0, $2n^{n/2-2}$ |
| $n-3$ | 0, $4n^{n/2-3}$ |
| $n-4$ | 0, $8n^{n/2-4}$, $16n^{n/2-4}$ |

TABLE 1.2
*Values of minors for Hadamard matrices of orders 12 and 16*

| minor | values of minors |
|---|---|
| $n-5$ | 0, $16n^{n/2-5}$, $32n^{n/2-5}$, $48n^{n/2-5}$ |
| $n-6$ | 0, $32n^{n/2-6}$, $64n^{n/2-6}$, $96n^{n/2-6}$, $128n^{n/2-6}$, $160n^{n/2-6}$ |
| $n-7$ | 0, $64n^{n/2-7}$, $128n^{n/2-7}$, $192n^{n/2-7}$, $256n^{n/2-7}$, $320n^{n/2-7}$, $384n^{n/2-7}$, $448n^{n/2-7}$, $512n^{n/2-7}$, $576n^{n/2-7}$ |

A *Hadamard matrix* $H$ of order $n$ is a matrix with elements $\pm 1$ satisfying the orthogonality relation $HH^T = H^TH = nI_n$. It can be proved [11, 4] that if $H$ is a Hadamard matrix of order $n$ then $n = 1, 2$ or $n \equiv 0 \pmod 4$. However it is still an open conjecture whether Hadamard matrices exist for every $n$ being a multiple of 4. For more details on Hadamard matrices the reader can consult [8, 14, 22, 25].

The first known effort for calculating minors of Hadamard matrices was accomplished in [23] for the $n-1$, $n-2$ and $n-3$ minors. In [17] all possible $n-4$ minors of Hadamard matrices were calculated theoretically by a method that could be generalized as an algorithm.

The general results for the $n-j$ minors, $j = 1, \ldots, 4$, of Hadamard matrices are summarized in Table 1.1. The values for $j = 5, \ldots, 7$ are proved only for $n = 12$ and 16 [20], due to computational difficulties of the existing methods, and are given in Table 1.2. We see that all possible values of the $n-j$ minors, $j = 1, \ldots, 7$, follow a specific principle. This observation constitutes the following open conjecture for the possible values of minors of Hadamard matrices.

**The conjecture for minors of Hadamard matrices**

All possible $(n-j) \times (n-j)$, $j \geq 1$, minors of Hadamard matrices are

$$0 \text{ or } p \cdot n^{(n/2)-j}, \text{ for } p = 2^{j-1}, 2 \cdot 2^{j-1}, 3 \cdot 2^{j-1}, \ldots, s \cdot 2^{j-1},$$

where

$$s \cdot 2^{j-1} = \max\{\det(A) | A \in \mathbb{R}^{j \times j}, \text{with entries } \pm 1\}$$

and the value 0 is excluded from the case $j = 1$.

The maximum determinant values for $\pm 1$ matrices are given in Table 1.3. The study of the above conjecture might furthermore lead to useful results concerning the possible values of determinants of $\pm 1$ matrices, which are not exactly specified even for relatively small orders ($n = 8$). The relevant known results are given in [4, 19].

A *binary Hadamard matrix* (called also *S-matrix*) is an $n \times n$ $(0, 1)$ matrix formed by taking an $(n+1) \times (n+1)$ Hadamard matrix in which the entries in the first row and column are $+1$, changing $+1$'s to 0's and $-1$'s to $+1$'s, and deleting the

TABLE 1.3
*Maximum determinants of ±1 matrices*

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| max. det. | 1 | 2 | 4 | 16 | 48 | 160 | 576 |

first row and column. Therefore, $n \equiv 3 \pmod 4$. A binary Hadamard matrix satisfies $SS^T = S^T S = \frac{1}{4}(n+1)(I_n + J_n)$ and $SJ_n = J_n S = \frac{1}{2}(n+1)J_n$. Further information on binary Hadamard matrices, their applications and related problems can be found in [12, 8, 22, 25] and in the references therein.

*Remark.* It is important to emphasize that the present work deals only with the specific binary Hadamard matrices that are obtained from the cores of Hadamard matrices according to the construction described above. These binary Hadamard matrices are actually the incidence matrices of symmetric balanced incomplete block designs (SBIBDs) with parameters $(4t-1, 2t, t)$ [5, 8]. Indeed, if $H$ is a Hadamard matrix of order $4t$, then it can be written in the form

$$H = \begin{bmatrix} 1 & e^T \\ e & A \end{bmatrix},$$

where $e^T = (1, 1, \ldots, 1)$ is the $1 \times (4t-1)$ vector with elements 1. Then the matrix $C = \frac{1}{2}(J_{4t-1} - A)$ is the incidence matrix of an SBIBD with parameters $(4t-1, 2t, t)$. A SBIBD with parameters $(4t-1, 2t, t)$ is the complement of an SBIBD with parameters $(4t-1, 2t-1, t-1)$, the incidence matrix of which is constructed as $\frac{1}{2}(A + J_{4t-1})$. In [18] values of minors for various families of $(1, -1)$ incidence matrices of SBIBDs were computed and the growth problem was discussed for them.

*Example.* Binary Hadamard matrices of small orders.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

**1.2. Notations and preliminary remarks.** *Notations.* Wherever determinant or minor is mentioned in this work, we mean its magnitude, i.e. the absolute value. $I_n$ and $J_n$ stand for the identity matrix and the matrix with ones of order $n$, respectively. Whenever information on the dimension is not needed, the indices are omitted. We write $A(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper left corner of the matrix $A$, i.e. $A(j)$ is the magnitude of the $j \times j$ leading principal minor of $A$. An $m \times n$ matrix having all its entries equal to $x \in \mathbb{R}$ will be denoted by $x_{m \times n}$. If $x$ consists of more than one terms (e.g. $x = k-1$), then parentheses are used around $x$ for avoiding confusion.

The designation $(\kappa - \lambda)I + \lambda J$ for describing briefly a matrix of the form

$$\begin{bmatrix} \kappa & \lambda & \cdots & \lambda \\ \lambda & \kappa & \cdots & \lambda \\ \vdots & & \ddots & \\ \lambda & \lambda & \cdots & \kappa \end{bmatrix}$$

will be used frequently. If it is necessary to specify the order $n$ of such a matrix $X = (\kappa - \lambda)I + \lambda J$, then it will be denoted by $X_{n\times n} \equiv (\kappa - \lambda)I_n + \lambda J_n$. So, for instance, $X_{2\times 2} = [\begin{smallmatrix}\kappa & \lambda \\ \lambda & \kappa\end{smallmatrix}]$.

Let $\underline{y}^T_{\beta+1}$ be the vectors containing the binary representation of each integer $\beta = 2^j - 1, \dots, 0$. Define the $j \times 1$ vectors $\underline{u}_k = \underline{y}_{2^j-k+1}$, $k = 1, \dots, 2^j$. We write $U_j$ for all the matrices with $j$ rows and the appropriate number of columns, in which $\underline{u}_k$ occurs $u_k$ times. So

$$U_j = \begin{array}{ccccc} \overbrace{1...1}^{u_1} & \overbrace{1...1}^{u_2} & \dots & \overbrace{0...0}^{u_{2^j-1}} & \overbrace{0...0}^{u_{2^j}} \\ 1...1 & 1...1 & \dots & 0...0 & 0...0 \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ 1...1 & 1...1 & \dots & 0...0 & 0...0 \\ 1...1 & 0...0 & \dots & 1...1 & 0...0 \end{array} = \begin{array}{ccccc} u_1 & u_2 & \dots & u_{2^{j-1}-1} & u_{2^{j-1}} \\ 1 & 1 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 1 & 0 \end{array}.$$

*Example.*

$$U_3 = \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

$$U_4 = \begin{array}{cccccccccccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

The matrix $U_j$ is important in this study because it depicts a general form for $j$ arbitrary rows of a binary Hadamard matrix.

We provide some useful formulas for two specially structured matrices, which will be used repeatedly throughout the paper.

LEMMA 1.2. *Let $A = (\kappa - \lambda)I_v + \lambda J_v$, where $\kappa$, $\lambda$ are integers. Then,*

$$\det A = [\kappa + (v-1)\lambda](\kappa - \lambda)^{v-1} \tag{1.1}$$

*and for $\kappa \neq \lambda, -(v-1)\lambda$, $A$ is nonsingular with*

$$A^{-1} = \frac{1}{\kappa^2 + (v-2)\kappa\lambda - (v-1)\lambda^2}\{[\kappa + (v-1)\lambda]I_v - \lambda J_v\}. \tag{1.2}$$

Equation (1.2) is a special case of the Sherman-Morrison formula [3, p.239], which computes the inverse of a rank-one-correction of a nonsingular matrix B as

$$(B - uv^T)^{-1} = B^{-1} + \frac{B^{-1}uv^TB^{-1}}{1 - v^TB^{-1}u},$$

where $u, v$ are vectors and $v^TB^{-1}u \neq 1$. Indeed, (1.2) occurs for $B = (\kappa - \lambda)I_v$ and $u = -\lambda[1\ 1\ \dots\ 1]^T$ and $v = [1\ 1\ \dots\ 1]^T$.

LEMMA 1.3. (Schur determinant formula) [15, p.21] *Let* $B = \begin{array}{cc} B_1 & B_2 \\ B_3 & B_4 \end{array}$. *If $B_1$ is nonsingular, then*

$$\det B = \det B_1 \cdot \det(B_4 - B_3B_1^{-1}B_2). \tag{1.3}$$

*If $B_4$ is nonsingular, then*

$$\det B = \det B_4 \cdot \det(B_1 - B_2 B_4^{-1} B_3). \tag{1.4}$$

The paper is organized as follows. In §2 the strategy for computing all possible $n-j$ minors of binary normalized orthogonal matrices is outlined. In §3 we give helpful lemmas and results for minors of binary Hadamard matrices. An algorithm for this purpose is also presented. The growth problem is generally described in §4, the growth conjecture for binary Hadamard matrices is formulated and information about pivot patterns of binary Hadamard matrices is given. Finally, §5 summarizes the results of this work and highlights further improvements and possibilities.

**2. The numerical technique for the evaluation of minors.** Next we describe the proposed technique for calculating all possible $(n-j)\times(n-j)$ minors of binary normalized orthogonal matrices of order $n$ in a general context. However, it can be understood better through the theoretical proofs and the algorithm of §3.

In order to calculate all possible $(n-j)\times(n-j)$ minors of a binary Hadamard matrix $S$, we write it in the form

$$S = \begin{bmatrix} M & U_j \\ V_j^T & D \end{bmatrix},$$

where $M$ and $D$ are quadratic matrices of orders $j$ and $n-j$, respectively. The purpose is to write down, for every possible upper left $j\times j$ corner $M$, the values appearing for the determinant of $D$, which is actually the required minor. The submatrices $U_j$ and $V_j^T$ contain all possible $j\times(n-j)$ columns and $(n-j)\times j$ rows, respectively, which can appear in $S$, so they must be created very carefully according to the properties of $A$. Although, for some binary Hadamard matrices, with appropriate row and/or column interchanges and, if necessary, with multiplications by $-1$ $U_j = V_j$ can be achieved, in general it cannot. It is very important that same columns are clustered together in $U_j$, as in §1.2. In this manner, the computations are facilitated by the block forms appearing and the derivation of analytical formulas is possible.

From the order of $S$, from the inner product of its first $j$ rows and, if necessary, from the total number of 1's and 0's in the first $j$ rows of $S$ we set up and solve a system with unknowns the numbers of columns of $U_j$. If an exact solution can be found (i.e. there don't exist parameters in the solution), the method provides a general formula. If parameters exist in the solution, we must determine upper bounds (depending on $n$) that give all feasible values for the parameters. So, in this case the result will be not general but dependent on $n$.

Afterwards we evaluate $D^T D$ (or equivalently $DD^T$) taking into account $S^T S = \frac{1}{4}(n+1)(I_n + J_n)$ (or equivalently $SS^T = \frac{1}{4}(n+1)(I_n+J_n)$) and write the result appropriately in block form. The sizes of the blocks are known since they correspond to the solution of the system of equations described above. Finally we aim at deriving $\det D^T D$ by consecutive applications of formula (1.3) or (1.4), with help of (1.1) and (1.2).

Due to the properties of $S$, all diagonal blocks of $D^T D$ will be of the form $(a-b)I + bJ$ and each of the rest of the blocks consists only of a constant. Also, $D^T D$ is always symmetric. These properties are taken always into account during the computations, so that every matrix multiplication, inversion and determinant evaluation is not performed explicitly but in an efficient manner with help of (1.1), (1.2), (1.3) and (1.4).

It is important to emphasize that the proposed method calculates all possible $(n-j)\times(n-j)$ minors. The selected rows, which are written as the first $j$ rows of $A$, do not necessarily appear there but can be located anywhere in the matrix. They can be made to appear as first rows with appropriate row and/or column interchanges. We write them at the top only for the sake of better presentation and without any loss of generality. The fact that we examine *all possible upper left $j\times j$ corners* guarantees that with this technique we calculate *all possible $(n-j)\times(n-j)$ minors of $A$* and that we don't miss out any values which might appear.

It is also important to stress that the method can be very easily modified to work for Hadamard matrices as well. We believe it is sensible to present it through the example of binary Hadamard matrices, which represents a comprehensive but not extensive range of calculations and yields new results.

The method can be implemented in a Computer Algebra System, like Maple, in a symbolical sense, which guarantees the accuracy of the result avoiding any roundoff errors and preserves analytical formulas.

**3. Main results.** We demonstrate theoretical formulas for the $n-j$ minors, $j=1,2,3,4$, of binary Hadamard matrices. We explain why the proof of results for $j>2$ is more complicated and why it can be carried out with this technique only for specific, fixed values of $n$. First we give the following, almost straightforward Lemma. In the proofs of this section we set $k:=\frac{1}{4}(n+1)$.

LEMMA 3.1. *The determinant of a binary Hadamard matrix $S$ of order $n$ is $2^{-n}(n+1)^{\frac{n+1}{2}}$.*

*Proof.* From the definition of $S$ we have

$$SS^T = k(I_n+J_n) = k\begin{bmatrix} 2 & 1 & \cdots \\ 1 & 2 & \\ \vdots & & \ddots \end{bmatrix} = k[(2-1)I_n+J_n].$$

Equation (1.1) gives

$$\det SS^T = k^n[2+(n-1)] = k^n(n+1) = \frac{(n+1)^{n+1}}{4^n}.$$

Since $\det SS^T = (\det S)^2$, we have

$$|\det S| = \frac{(n+1)^{\frac{n+1}{2}}}{2^n},$$

which proves the assumption. $\square$

*Remark.* Lemma 3.1 can be also proved by using the explicit connection between a Hadamard matrix $H$ of order $n+1$ and a binary Hadamard matrix $S$ of order $n$, namely $H-J_{n+1} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & -2S \end{smallmatrix}\right]$. This idea takes advantage of the invertibility of any $n\times n$ submatrix of $H$, since all such minors are nonzero, cf. Table 1.1, while it doesn't seem to be applicable for deriving more results on minors of $S$, since invertibility cannot be guaranteed for submatrices of $H$ of smaller orders.

Before proceeding to further computations, we need the following Lemma 3.2, which gives the number of 1's and 0's in every row and column of a binary Hadamard matrix.

Lemma 3.2. *Every row and every column of a binary Hadamard matrix $S$ of order $n$ has $\frac{n+1}{2}$ 1's and $\frac{n-1}{2}$ 0's.*

*Proof.* The result is derived from the property $SJ_n = J_nS = \frac{1}{2}(n+1)J_n$, which shows that the sum of the entries of every row and column of a binary Hadamard matrix is $\frac{n+1}{2}$, considering that the entries of the matrix are only (0,1). ◻

Proposition 3.3. *Let $S$ be a binary Hadamard matrix of order $n$. Then all possible $(n-1)\times(n-1)$ minors of $S$ are $2^{1-n}(n+1)^{\frac{n-1}{2}}$.*

*Proof.* Since $S$ is a binary Hadamard matrix of order $n$, it can be written in one of the following two forms:

$$S = \left[\begin{array}{c|c} 1 & \overbrace{1 \ \dots \ 1}^{(n-1)/2} \quad \overbrace{0 \ \dots \ 0}^{(n-1)/2} \\ \hline 1 & \\ \vdots & \\ 1 & \mathrm{A} \\ 0 & \\ \vdots & \\ 0 & \end{array}\right] \text{ or } \left[\begin{array}{c|c} 0 & \overbrace{1 \ \dots \ 1}^{(n+1)/2} \quad \overbrace{0 \ \dots \ 0}^{(n-3)/2} \\ \hline 1 & \\ \vdots & \\ 1 & \mathrm{A}' \\ 0 & \\ \vdots & \\ 0 & \end{array}\right],$$

where the first columns contain the appropriate number of 1's and 0's below the horizontal line, so that they have $\frac{n+1}{2}$ 1's and $\frac{n-1}{2}$ 0's.
From the definition of $S$, $S^TS = k(I_n + J_n)$, it follows that the $(n-1)\times(n-1)$ matrix $A^TA$ has the form

$$A^TA = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_3 \end{bmatrix},$$

where

$$\begin{aligned} A_1 &= [2k-1-(k-1)]I_{\frac{n-1}{2}} + (k-1)J_{\frac{n-1}{2}}, \\ A_2 &= kJ_{\frac{n-1}{2}} \quad \text{and} \\ A_3 &= (2k-k)I_{\frac{n-1}{2}} + kJ_{\frac{n-1}{2}}. \end{aligned}$$

From (1.4) we have

$$\det A^TA = \det A_3 \cdot \det(A_1 - A_2A_3^{-1}A_2). \tag{3.1}$$

Equation (1.1) gives

$$\det A_3 = 2^{-n}(n+1)^{\frac{n+1}{2}}. \tag{3.2}$$

With help of (1.2) we obtain

$$A_3^{-1} = \frac{1}{k(n+1)}\{[(n-1)+2]I_{\frac{n-1}{2}} - 2J_{\frac{n-1}{2}}\},$$

and careful calculations give

$$A_1 - A_2A_3^{-1}A_2 = \frac{1}{4}\{[(n-1)+2]I_{\frac{n-1}{2}} - 2J_{\frac{n-1}{2}}\}.$$

Equation (1.1) yields

$$\det(A_1 - A_2A_3^{-1}A_2) = 2^{2-n}(n+1)^{\frac{n-3}{2}}. \tag{3.3}$$

Substitution of (3.2) and (3.3) in (3.1) gives $\det A^T A = 2^{2-2n}(n+1)^{n-1}$, hence

$$|\det A| = 2^{1-n}(n+1)^{\frac{n-1}{2}}.$$

From this proof it becomes obvious that we obtain the same value for $\det A$, independently of the possible relative position of $A$ inside $S$, if $S$ is compelled to be in the first possible form.
Working similarly for the second possible form of $S$ yields the same result for $\det A'$. Hence, we conclude that all possible $(n-1)\times(n-1)$ minors of $S$ are of magnitude $2^{1-n}(n+1)^{\frac{n-1}{2}}$. ◻

Proposition 3.4. *Let $S$ be a binary Hadamard matrix of order $n$, $n > 2$. Then all possible $(n-2)\times(n-2)$ minors of $S$ are 0 or $2^{3-n}(n+1)^{\frac{n-3}{2}}$.*

*Proof.* There are $2^4$=16 possible cases for the upper left $2\times 2$ corner of $S$, since there are two possible entries, 0 and 1. We will illustrate the proof for one matrix, since the other cases can be handled absolutely similarly. Since we have that $S$ is an $n\times n$ binary Hadamard matrix, let us suppose that it can be written in the following form:

$$S = \left[\begin{array}{cc|cccc}
1 & 1 & \overbrace{1\ \dots\ 1}^{u} & \overbrace{1\ \dots\ 1}^{v} & \overbrace{0\ \dots\ 0}^{x} & \overbrace{0\ \dots\ 0}^{y} \\
1 & 0 & 1\ \dots\ 1 & 0\ \dots\ 0 & 1\ \dots\ 1 & 0\ \dots\ 0 \\ \hline
1 & 1 & & & & \\
\vdots & \vdots & & & & \\
1 & 1 & & & & \\
1 & 0 & & & & \\
\vdots & \vdots & & & & \\
1 & 0 & & & \mathrm{B} & \\
0 & 1 & & & & \\
\vdots & \vdots & & & & \\
0 & 1 & & & & \\
0 & 0 & & & & \\
\vdots & \vdots & & & & \\
0 & 0 & & & &
\end{array}\right].$$

From the order of the matrix $S$, the inner product of its first two rows and from the total number of 0's in the first two rows (according to Lemma 3.2), we get the following system of four equations

$$\begin{cases} u+v+x+y = n-2 \\ 1+u = \frac{n+1}{4} \\ x+y = \frac{n-1}{2} \\ 1+v+y = \frac{n-1}{2} \end{cases},$$

which has the exact solution

$$(u,v,x,y) = \frac{1}{4}(n-3, n-3, n+1, n-3).$$

According to the properties of an $n\times n$ binary Hadamard matrix, the $(n-2)\times(n-2)$

matrix $B^TB$ has the form

$$B^TB = \left[\begin{array}{c|ccc} B_{1u\times u} & (k-1)_{u\times v} & (k-1)_{u\times x} & k_{u\times y} \\ \hline (k-1)_{v\times u} & B_{2v\times v} & k_{v\times x} & k_{v\times y} \\ (k-1)_{x\times u} & k_{x\times v} & B_{2x\times x} & k_{x\times y} \\ k_{y\times u} & k_{y\times v} & k_{y\times x} & B_{3y\times y} \end{array}\right] \equiv \left[\begin{array}{cc} B_{1u\times u} & F \\ F^T & G \end{array}\right],$$

where

$$\begin{aligned} &B_1 = [(2k-2)-(k-2)]I + (k-2)J, \\ &B_2 = [(2k-1)-(k-1)]I + (k-1)J \quad \text{and} \\ &B_3 = (2k-k)I + kJ. \end{aligned}$$

So, according to (1.3),

$$\det B^TB = \det B_{1u\times u} \cdot \det(G - F^T B_{1u\times u}^{-1} F). \tag{3.4}$$

From (1.1) we have

$$\det B_{1u\times u} = \frac{\sqrt{2}}{2}(n^2 - 6n + 25)\left(\frac{n+1}{4}\right)^{\frac{n}{4}}(n+1)^{-\frac{7}{4}} \tag{3.5}$$

and from (1.2) $B_{1u\times u}^{-1} = (k_1 - \lambda_1)I_u + \lambda_1 J_u$, where $k_1 = \frac{4(n^2-10n+53)}{(n-1)(n^2-6n+25)}$ and $\lambda_1 = \frac{16(n-7)}{(n-1)(n^2-6n+25)}$.
Hence,

$$G - F^T B_{1u\times u}^{-1} F = \left[\begin{array}{cc} K_{1v\times v} & N_2 \\ N_2^T & N_1 \end{array}\right],$$

where the blocks $K_{1v\times v}$, $N_1$ and $N_2$ are calculated. For the sake of brevity and better presentation it is not sensible to give all the intermediate matrices analytically, however they are available from the authors on request.
From (1.3) we have

$$\det(G - F^T B_{1u\times u}^{-1} F) = \det K_{1v\times v} \cdot \det(N_1 - N_2^T {K_{1v\times v}}^{-1} N_2). \tag{3.6}$$

From this point and on, the idea of the proof is to apply consecutively formula (1.3) appropriately for the block matrices appearing and carry out the calculations with the help of (1.1) and (1.2).

We proceed absolutely similarly as before in order to calculate $\det K_{1v\times v}$ and $\det(N_1 - N_2^T K_{v\times v}^{-1} N_2)$, by making use of (1.1), (1.2) and (1.3).
We have

$$\det K_{1v\times v} = \frac{2\sqrt{2}(n^3 - n^2 - 5n + 61)}{n^2 - 6n + 25}\left(\frac{n+1}{4}\right)^{\frac{n}{4}}(n+1)^{-\frac{7}{4}} \tag{3.7}$$

and

$$N_1 - N_2^T {K_{1v\times v}}^{-1} N_2 = \left[\begin{array}{cc} P_{1x\times x} & Q_1 \\ Q_1^T & P_{2y\times y} \end{array}\right],$$

where the blocks $P_{1x\times x}$, $Q_1$ and $P_{2y\times y}$ are obtained as described.

According to (1.3),

$$\det(N_1 - N_2^T K_{1v\times v}{}^{-1} N_2) = \det P_{1x\times x}\ \det(P_{2y\times y} - Q_1^T P_{1x\times x}{}^{-1} Q_1). \tag{3.8}$$

From (1.1) and (1.2) we have

$$\det P_{1x\times x} = \frac{2\sqrt{2}(n+1)^{\frac{1}{4}}(n^2-2n+13)(\frac{n+1}{4})^{\frac{n}{4}}}{n^3-n^2-5n+61} \tag{3.9}$$

and

$$P_{2y\times y} - Q_1^T P_{1x\times x}{}^{-1} Q_1 = R_{3y\times y}, \tag{3.10}$$

where $R_{3y\times y} = (k_2-\lambda_2)I_y + \lambda_2 J_y$, $k_2 = \frac{n^3-5n^2+19n+25}{4(n^2-2n+13)}$ and $\lambda_2 = -\frac{n^2-2n-3}{n^2-2n+13}$.
Equation (1.1) gives

$$\det R_{3y\times y} = \frac{8\sqrt{2}(n+1)^{\frac{1}{4}}(\frac{n+1}{4})^{\frac{1}{4}}}{n^2-2n+13}. \tag{3.11}$$

Finally, from (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11) we have

$$\det B^T B = \det E_{u\times u}\ \det K_{1v\times v}\ \det P_{1x\times x}\ \det R_{3y\times y} = \frac{64(\frac{n+1}{4})^n}{(n+1)^3} = 4^{3-n}(n+1)^{n-3}.$$

Hence, $|\det B| = 2^{3-n}(n+1)^{\frac{n-3}{2}}$.
Similarly we handle all possible remaining cases for the upper left hand corner and obtain the same result and the value 0. ∎

The following Lemmas 3.5 and 3.6 specify the possible number of columns of a binary Hadamard matrix if only few rows of it are considered. They are useful for carrying out proofs like the one of Proposition 3.4 for $(n-j)\times(n-j)$ minors, $j>2$. In such cases, the linear systems occurring from the properties of binary Hadamard matrices contain parameters and cannot be solved exactly. Lemma 3.6 can be used to establish bounds for the parameters in the solutions of the systems, which actually represent columns of a binary Hadamard matrix, if the first $j$ rows are considered separately. Hence, there exist constraints on the number of columns of a binary Hadamard matrix, which moreover limit the calculations of the proposed technique. Since the upper bounds for the parameters are dependant on the order $n$, we cannot provide general results in these cases, but only for specific $n$ fixed.

Lemma 3.5. *Let $S$ be a binary Hadamard matrix of order $n$, $n>2$. Then for every triple of rows of $S$ the number of columns which are*
*(a)* $(1,1,1)^T$ *or* $(0,0,0)^T$ *is* $(n-3)/4$,
*(b)* $(1,1,0)^T$ *or* $(0,0,1)^T$ *is* $(n+1)/4$,
*(c)* $(1,0,1)^T$ *or* $(0,1,0)^T$ *is* $(n+1)/4$,
*(d)* $(1,0,0)^T$ *or* $(0,1,1)^T$ *is* $(n+1)/4$.

*Proof.* Three rows of the binary Hadamard matrix $S$ can be written as

$$\begin{array}{cccccccc}
\overbrace{\begin{matrix}1 & \dots & 1\end{matrix}}^{u_1} & \overbrace{\begin{matrix}1 & \dots & 1\end{matrix}}^{u_2} & \overbrace{\begin{matrix}1 & \dots & 1\end{matrix}}^{u_3} & \overbrace{\begin{matrix}1 & \dots & 1\end{matrix}}^{u_4} & \overbrace{\begin{matrix}0 & \dots & 0\end{matrix}}^{u_5} & \overbrace{\begin{matrix}0 & \dots & 0\end{matrix}}^{u_6} & \overbrace{\begin{matrix}0 & \dots & 0\end{matrix}}^{u_7} & \overbrace{\begin{matrix}0 & \dots & 0\end{matrix}}^{u_8} \\
1 \ \dots \ 1 & 1 \ \dots \ 1 & 0 \ \dots \ 0 & 0 \ \dots \ 0 & 1 \ \dots \ 1 & 1 \ \dots \ 1 & 0 \ \dots \ 0 & 0 \ \dots \ 0 \\
1 \ \dots \ 1 & 0 \ \dots \ 0 & 1 \ \dots \ 1 & 0 \ \dots \ 0 & 1 \ \dots \ 1 & 0 \ \dots \ 0 & 1 \ \dots \ 1 & 0 \ \dots \ 0
\end{array}.$$

From the order of the matrix $S$, the inner product of its first three rows and from the total number of 1's and 0's in the first three rows (according to Lemma 3.2), we get the following system of ten equations

$$\begin{cases} u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8=n \\ u_1+u_2=\frac{n+1}{4} \\ u_1+u_3=\frac{n+1}{4} \\ u_1+u_5=\frac{n+1}{4} \\ u_1+u_2+u_3+u_4=\frac{n+1}{2} \\ u_5+u_6+u_7+u_8=\frac{n-1}{2} \\ u_1+u_2+u_5+u_6=\frac{n+1}{2} \\ u_3+u_4+u_7+u_8=\frac{n-1}{2} \\ u_1+u_3+u_5+u_7=\frac{n+1}{2} \\ u_2+u_4+u_6+u_8=\frac{n-1}{2} \end{cases},$$

which has the solution

$$\begin{array}{l} u_1=\frac{n-3}{4}-u_8 \\ u_2=u_8+1 \\ u_3=u_8+1 \\ u_4=\frac{n-3}{4}-u_8 \\ u_5=u_8+1 \\ u_6=\frac{n-3}{4}-u_8 \\ u_7=\frac{n-3}{4}-u_8 \\ u_8=u_8 \end{array}.$$

From the solution we see that

$$\begin{array}{l} u_1+u_8=\frac{n-3}{4}, \\ u_2+u_7=u_3+u_6=u_4+u_5=\frac{n+1}{4}, \end{array}$$

and the result follows obviously. $\square$

Lemma 3.6. *Let $S$ be a binary Hadamard matrix of order $n$, $n>2$. For all the $2^j$ possible columns $\underline{u}_1,\ldots,\underline{u}_{2^j}$ of $S$ (or $U_j$) comprising the first $j$ rows, $j\geq 3$, it holds*

$$0\leq u_i\leq \frac{n-3}{4}\ ,\ for\ \ i\in\left\{1,\ldots,\frac{1}{8}\cdot 2^j\right\}\cup\left\{\frac{7}{8}\cdot 2^j+1,\ldots,2^j\right\}$$

*and*

$$0\leq u_i\leq\frac{n+1}{4}\ ,\ otherwise.$$

*Proof.* If we consider separately the first three rows from the first $j$ rows of the enunciation, we observe for the $2^j$ possible columns $\underline{u}_i,\ i=1,\ldots,2^j$, of $U_j$, that

$$\begin{array}{rcccccl} \underline{u}_1(1:3) &=& \ldots &=& \underline{u}_{\frac{1}{8}2^j}(1:3) &=& (1,1,1)^T \\ \underline{u}_{\frac{1}{8}2^j+1}(1:3) &=& \ldots &=& \underline{u}_{\frac{2}{8}2^j}(1:3) &=& (1,1,0)^T \\ \underline{u}_{\frac{2}{8}2^j+1}(1:3) &=& \ldots &=& \underline{u}_{\frac{3}{8}2^j}(1:3) &=& (1,0,1)^T \\ \underline{u}_{\frac{3}{8}2^j+1}(1:3) &=& \ldots &=& \underline{u}_{\frac{4}{8}2^j}(1:3) &=& (1,0,0)^T \\ \underline{u}_{\frac{4}{8}2^j+1}(1:3) &=& \ldots &=& \underline{u}_{\frac{5}{8}2^j}(1:3) &=& (0,1,1)^T \\ \underline{u}_{\frac{5}{8}2^j+1}(1:3) &=& \ldots &=& \underline{u}_{\frac{6}{8}2^j}(1:3) &=& (0,1,0)^T \\ \underline{u}_{\frac{6}{8}2^j+1}(1:3) &=& \ldots &=& \underline{u}_{\frac{7}{8}2^j}(1:3) &=& (0,0,1)^T \\ \underline{u}_{\frac{7}{8}2^j+1}(1:3) &=& \ldots &=& \underline{u}_{\frac{8}{8}2^j}(1:3) &=& (0,0,0)^T \end{array},$$

where $\underline{u}_i(1:3)$ denotes the first three entries (like in Matlab notation) of the column $\underline{u}_i$ of $U_j$. This observation arises easily from a combinatorial counting and can be also verified with the matrices $U_3$ and $U_4$ given in §1.2.

From Lemma 3.5 we conclude

$$\begin{array}{ccccccccccccc} u_1 & + & \dots & + & u_{\frac{1}{8}2^j} & + & u_{\frac{7}{8}2^j+1} & + & \dots & + & u_{\frac{8}{8}2^j} & = & \frac{n-3}{4} \\ u_{\frac{1}{8}2^j+1} & + & \dots & + & u_{\frac{2}{8}2^j} & + & u_{\frac{6}{8}2^j+1} & + & \dots & + & u_{\frac{7}{8}2^j} & = & \frac{n+1}{4} \\ u_{\frac{2}{8}2^j+1} & + & \dots & + & u_{\frac{3}{8}2^j} & + & u_{\frac{5}{8}2^j+1} & + & \dots & + & u_{\frac{6}{8}2^j} & = & \frac{n+1}{4} \\ u_{\frac{3}{8}2^j+1} & + & \dots & + & u_{\frac{4}{8}2^j} & + & u_{\frac{4}{8}2^j+1} & + & \dots & + & u_{\frac{5}{8}2^j} & = & \frac{n+1}{4} \end{array} .$$

The result follows straightforwardly from these relations by taking into account that $u_i \geq 0$ since $u_i$ denote number of columns. $\square$

In addition, another difficulty in calculating $(n-j) \times (n-j)$ minors, $j > 2$, arises from the fact that we have to examine $2^{j^2}$ possible upper left corners, which are singled out from the general form of $j$ rows of a binary Hadamard matrix denoted by $U_j$. This observation in combination with the fact that the calculations in the proofs of Propositions 3.3 and 3.4 follow a predictable, standard procedure based on successive applications of formula (1.3) or (1.4), led us to develop this technique from an algorithmic point of view. So, there was constructed the following algorithm Minors, which is intended for calculating all possible $(n-j) \times (n-j)$ minors, $j \geq 1$, of binary Hadamard matrices.

The implementation on the computer starts by letting the computer algebra package have as input every possible upper left corner. For each one of them, the necessary computations are performed in a symbolical sense. Algorithm Minors can be applied theoretically for every values $n$ and $j$. Lemma 3.6 is used for finding the possible values for the parameters in the solutions of the appearing linear systems. In the following algorithm the notation $V_j$ stands for all possible columns with entries 0 and 1, like $U_j$, but we choose another letter to show that the matrices $U_j$ and $V_j$ are not necessarily the same.

**Algorithm Minors**

**Input:** All possible $j \times j$ matrices $M$, which can exist in the upper left corner of an $n \times n$ binary Hadamard matrix $S = \begin{bmatrix} M & U_j \\ V_j^T & D \end{bmatrix}$.

**Output:** Absolute values of all possible $(n-j) \times (n-j)$ minors of $S$.

**Step 1:** *FOR EVERY* matrix $M$

*FORM* the system of $1 + \binom{j}{2} + 2j$ equations and $2^j$ unknowns $u_i$ that results by counting of columns, by the inner products of every two distinct rows of the matrix $[M \;\; U_j]$ and by the total number of 1's and 0's in every row of $[M \;\; U_j]$.

*SOLVE* the system for all $u_i$.

**Step 2:** *FOR* all the parameters attaining the values $0, \dots, \frac{n-3}{4}$ or $\frac{n+1}{4}$

**Step 3:** *IF* $u_i \geq 0$ and $u_i$ integers, $i = 1, \dots, 2^j$

$$D^T D \equiv \begin{bmatrix} E_1 & F_1 \\ F_1^T & G_1 \end{bmatrix}$$

$$G_1 - F_1^T E_1^{-1} F_1 \equiv \begin{bmatrix} E_2 & F_2 \\ F_2^T & G_2 \end{bmatrix}$$

*ELSE* there are no acceptable solutions

*END IF*

**Step 4:** *FOR* $k = 2, \dots, 2^j - 2$

$$G_k - F_k^T E_k^{-1} F_k \;\equiv \left[ \begin{array}{cc} E_{k+1} & F_{k+1} \\ F_{k+1}^T & G_{k+1} \end{array} \right]$$

*END*

**Step 5:** $E_{2^j} = G_{2^j-1} - F_{2^j-1}^T E_{2^j-1}^{-1} F_{2^j-1}$

**Step 6:** $\det D^T D := \prod_{i=1}^{2^j} \det E_i, \;\; |\det D| = \sqrt{\det D^T D}$

*END* {for all the parameters }

*END* {for every matrix $M$ }

*END* {of Algorithm}

PROPOSITION 3.7. *Let $S$ be a binary Hadamard matrix of order $n = 11$. Then all possible $(n-3)\times(n-3)$ minors of $S$ are 0 or $2^{5-n}(n+1)^{\frac{n-5}{2}}$.*

*Proof.* The idea is similar to the proof of Proposition 3.4. $S$ is written in the form

$$S = \left[ \begin{array}{cc} M & U_3 \\ V_3^T & D \end{array} \right],$$

and all possible $2^9 = 512$ $3\times 3$ upper left corners $M$ are taken as input for algorithm Minors. The familiar properties of $S$ lead to a linear system, which has the same left hand side as the system in the proof of Lemma 3.5, but different right hand sides, according to the upper left hand corner selected for $M$. The solutions $u_1, \dots, u_8$ representing the numbers of columns of $S$ are expressed in terms of the parameter $u_8$, which is allowed to take the values $0, 1, 2$ according to Lemma 3.6. For the acceptable solutions having positive integer components, the rest of the procedure of algorithm Minors is carried out in order to specify the determinant of $D$ for each $M$. For instance, at Step 3 of the algorithm, the matrix $D^T D$ has the form

$$D^T D = \left[ \begin{array}{cccccccc} E_1 & k_2 & k_2 & k_1 & k_2 & k_1 & k_1 & k \\ & F & k_1 & k_1 & k_1 & k_1 & k & k \\ & & F & k_1 & k_1 & k & k_1 & k \\ & & & G & k & k & k & k \\ & & & & F & k_1 & k_1 & k \\ & & & & & G & k & k \\ & & & & & & G & k \\ & & & & & & & H \end{array} \right],$$

where $k = \frac{n+1}{4}$, $k_1 = k-1$, $k_2 = k-2$, $E_1 = kI_{u_1} + (k-3)J_{u_1}$, $F = kI + (k-2)J$, $G = kI + (k-1)J$ and $H = kI + kJ$. The diagonal blocks $E_1$,$F$,$F$,$\dots$,$G$,$H$ are of known orders $u_1\times u_1$, $u_2\times u_2, \dots, u_8\times u_8$. The elements $k, k_1, k_2$ represent blocks with appropriate sizes (according to the notation in 1.2), but the subscripts are omitted for a more compact presentation. From now on all intermediate matrices result from the application of the algorithm Minors. For the sake of brevity we do not present all of them analytically. After all necessary evaluations are performed, the results given in the enunciation are obtained. ∎

In a similar manner, by applying algorithm Minors for $n = 11$ and $j = 4$, the following result is derived.

PROPOSITION 3.8. *Let $S$ be a binary Hadamard matrix of order $n = 11$. Then all possible $(n-4)\times(n-4)$ minors of $S$ are 0, $2^{7-n}(n+1)^{\frac{n-7}{2}}$ or $2^{8-n}(n+1)^{\frac{n-7}{2}}$.*

## 4. Application to the growth problem.

**4.1. Description of the problem.** Traditionally, backward error analysis for Gaussian Elimination (GE) on a matrix $A = (a_{ij}^{(0)})$ is expressed in terms of the *growth factor*

$$g(n, A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}^{(0)}|},$$

which involves all the elements $a_{ij}^{(k)}$, $k = 0, 1, 2, \ldots, n-1$ that occur during the elimination [3, 13, 24]. Matrices with the property that no row and column exchanges are needed during GE with complete pivoting are called *completely pivoted* (CP) or feasible. For a CP matrix $A$ we have

$$g(n, A) = \frac{\max\{p_1, p_2, \ldots, p_n\}}{|a_{11}^{(0)}|},$$

where $p_1, p_2, \ldots, p_n$ are the pivots of $A$. According to known theorems [3, 24], it is clear that the stability of GE depends on the growth factor. If $g(n, A)$ is of order 1, not much growth has taken place, and the elimination process is stable. If $g(n, A)$ is bigger than this, we must expect instability. The study of the values appearing for $g(n, A)$ and the specification of pivot patterns are referred to as *the growth problem.*

In [2] Cryer conjectured that "for real matrices $g(n, A) \leq n$, with equality if and only if $A$ is a Hadamard matrix". This conjecture became one of the most famous open problems in Numerical Analysis and has been investigated by many mathematicians. The inequality was finally shown to be false in [9], however its second part is still an open problem.

Since binary Hadamard matrices are connected with Hadamard matrices, it is sensible to apply GE with complete pivoting on equivalent (i.e. they are obtained by row and/or column interchanges) binary Hadamard matrices of various orders and write down their pivot patterns and growth factors.

Tables 4.1 and 4.2 show some pivot patterns which appear if GE with complete pivoting is applied, experimentally, to 200000 equivalent binary Hadamard matrices for each order $n = 15,\ 19,\ 23,\ 31$ and 39. The last column gives the total number of pivot patterns that have appeared in the experiments. Especially for $n = 15$, the binary Hadamard matrices are obtained separately from Hadamard matrices of order 16, which are classified in five equivalence classes I,II,...,V, see [25]. It is interesting to mention that the total numbers of pivot patterns observed experimentally for Hadamard matrices of order 16 from the equivalence classes I,II,...,V are 9,15,10,12,12, respectively. But for $n > 15$, the total numbers of pivot patterns occurring from binary Hadamard matrices are significantly fewer than the ones obtained from the corresponding Hadamard matrices, see [16]. The fourth pivot from the end is always $\frac{n+1}{8}$, except for binary Hadamard matrices of order 15 obtained from the I-equivalence class of Hadamard matrices of order 16. Finally, it is interesting to compare the pivot patterns of Hadamard and binary Hadamard matrices. We observe, cf. [16], that with the exception of the first pivot, which is in both cases always 1, the pivots of binary Hadamard matrices are about the half of the pivots of the corresponding Hadamard matrices, and consequently the growth factors are halved, too. This fact points out the significance for Gaussian elimination of inserting the entry 0 in binary Hadamard matrices instead of the entry $-1$ of Hadamard matrices.

These numerical experiments, in combination with the theoretical results of §3,

TABLE 4.1
*Pivot patterns of binary Hadamard matrices of order $n = 15$*

| class | pivot pattern | number |
|---|---|---|
| I | $(1, 1, 2, 1, \frac{4}{3}, 1, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$ | 12 |
| | $(1, 1, 2, 1, \frac{4}{3}, 2, 3, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$ | |
| | $(1, 1, 2, \frac{3}{2}, \frac{4}{3}, 1, 2, 2, 2, 2, 4, 4, 4, 4, 8)$ | |
| II | $(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$ | 15 |
| | $(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$ | |
| | $(1, 1, 2, 1, \frac{5}{3}, \frac{6}{5}, 2, \frac{8}{5}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$ | |
| III | $(1, 1, 2, 1, \frac{4}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$ | 18 |
| | $(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$ | |
| | $(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \frac{4}{3}, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$ | |
| IV/V | $(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 1, 2, 2, \frac{8}{3}, 2, 4, 4, 8)$ | 16 |
| | $(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 2, 2, \frac{12}{5}, \frac{8}{3}, 2, 4, 4, 8)$ | |
| | $(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, 2, \frac{20}{9}, \frac{12}{5}, \frac{8}{3}, 2, 4, 4, 8)$ | |

TABLE 4.2
*Pivot patterns of binary Hadamard matrices of orders $n = 19, 23, 31, 39$*

| n | pivot pattern | number |
|---|---|---|
| 19 | $(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \ldots, \frac{5}{2}, \frac{5}{2}, \frac{10}{3}, \frac{5}{2}, 5, 5, 10)$ | 187 |
| | $(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, \frac{9}{4}, \ldots, \frac{25}{9}, 3, 5, \frac{5}{2}, 5, 5, 10)$ | |
| | $(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, \frac{5}{2}, \ldots, \frac{25}{8}, \frac{15}{4}, 5, \frac{5}{2}, 5, 5, 10)$ | |
| 23 | $(1, 1, 2, 1, \frac{5}{3}, \frac{8}{5}, 2, \ldots, 3, 3, 4, 3, 6, 6, 12)$ | 228 |
| | $(1, 1, 2, \frac{3}{2}, \frac{5}{2}, \frac{9}{5}, 3, \ldots, \frac{10}{3}, \frac{18}{5}, 6, 3, 6, 6, 12)$ | |
| | $(1, 1, 2, \frac{3}{2}, 2, 2, 4, \ldots, \frac{15}{4}, \frac{9}{2}, 6, 3, 6, 6, 12)$ | |
| 31 | $(1, 1, 2, 1, \frac{5}{3}, \frac{8}{5}, 2, \ldots, 4, 4, \frac{16}{3}, 4, 8, 8, 16)$ | 595 |
| | $(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{5}{2}, \ldots, 4, 4, 8, 4, 8, 8, 16)$ | |
| | $(1, 1, 2, \frac{3}{2}, 2, 2, 3, \ldots, 4, 4, 8, 4, 8, 8, 16)$ | |
| 39 | $(1, 1, 2, 1, \frac{5}{3}, \frac{9}{5}, 2, \ldots, 5, 5, \frac{20}{3}, 5, 10, 10, 20)$ | 10000 |
| | $(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \ldots, 5, 5, \frac{20}{3}, 5, 10, 10, 20)$ | |
| | $(1, 1, 2, \frac{3}{2}, 2, 2, 3, \ldots, \frac{25}{4}, \frac{15}{2}, 10, 5, 10, 10, 20)$ | |

lead to the following new conjecture.

**The growth conjecture for binary Hadamard matrices**

Let $S$ be a binary Hadamard matrix of order $n$. Reduce $S$ by GE with complete pivoting. Then, for large enough $n$,

(i) $g(n, S) = \frac{n+1}{2}$;

(ii) Every pivot before the last has magnitude at most $\frac{n+1}{2}$;

(iii) The three last pivots are (in backward order) $\frac{n+1}{2}$, $\frac{n+1}{4}$, $\frac{n+1}{4}$;

(iv) The fourth pivot from the end can be $\frac{n+1}{8}$ or $\frac{n+1}{4}$;

(v) The first three pivots are equal to $1, 2, 2$. The fourth pivot can take the values 1 or 3/2.

**4.2. Pivot patterns of binary Hadamard matrices.** The object of this section it to demonstrate the unique pivot pattern of a CP binary Hadamard matrix of order 11, in other words to show that every equivalent binary Hadamard matrix can have only this one pivot pattern, if GE with complete pivoting is applied on it, or, equivalently, if GE is performed on a CP binary Hadamard matrix of order 11.

The challenge of this task is that a naive computer search finding all possible

binary Hadamard matrices of order 11 by performing all possible row and/or column interchanges would require $(11!)^2 \approx 10^{15}$ trials. This exhaustive search cannot be completed within a sensible period of time. In addition, the pivot pattern of each one of these matrices should be computed. Another obstacle when dealing with orders greater than 11 is the fact that the pivot pattern is not invariant up to equivalence row and/or column interchanges, i.e. it is possible that equivalent matrices can have different pivot patterns, cf. Table 4.1. Hence, in order to derive results about pivot patterns one cannot work with a representative matrix of an equivalence class of binary Hadamard matrices (e.g. the set of equivalent matrices with same determinant), which would limit the computations, but has to take all possible matrices into account. We show how the results of §3 can be utilized for calculating pivots from the end of the pivot structure in order to save significant computational time.

First, we give two useful properties for CP matrices.

LEMMA 4.1. [7, p.26], [21], [2] *Let $A$ be a CP matrix.*

(i) *The magnitude of the pivots appearing after application of GE operations on $A$ is given by*

$$p_j = \frac{A(j)}{A(j-1)}, \quad j = 1, 2, \dots, n, \quad A(0) = 1. \tag{4.1}$$

(ii) *The maximum $j \times j$ leading principal minor of $A$, when the first $j-1$ rows and columns are fixed, is $A(j)$.*

THEOREM 4.2. *If GE is applied on a CP binary Hadamard matrix of order $n$, $n > 2$, the last two pivots are (in backward order) $\frac{n+1}{2}$ and $\frac{n+1}{4}$.*

*Proof.* From Lemma 3.1 we have $\det S \equiv S(n) = 2^{-n}(n+1)^{\frac{n+1}{2}}$. Propositions 3.3 and 3.4, in combination with Lemma 4.1(ii), yield that for a CP binary Hadamard matrix hold $S(n-1) = 2^{1-n}(n+1)^{\frac{n-1}{2}}$ and $S(n-2) = 2^{3-n}(n+1)^{\frac{n-3}{2}}$. By substituting these values in relation (4.1) and taking into account Lemma 3.1, we obtain

$$p_n = \frac{S(n)}{S(n-1)} = \frac{2^{-n}(n+1)^{\frac{n+1}{2}}}{2^{1-n}(n+1)^{\frac{n-1}{2}}} = \frac{n+1}{2}$$

and

$$p_{n-1} = \frac{S(n-1)}{S(n-2)} = \frac{2^{1-n}(n+1)^{\frac{n-1}{2}}}{2^{3-n}(n+1)^{\frac{n-3}{2}}} = \frac{n+1}{4}.$$

∎

Considering the interpretation of a binary Hadamard matrix as SBIBD $(4t-1, 2t, t)$ we can state that the two last pivots (in backward order) are $2t$ and $t$. It is interesting to observe that the respective values for the complementary SBIBD $(4t-1, 2t-1, t-1)$ are $2t$ and $2t$ [18].

In the remainder of this paper we take advantage of the results of §3 for calculating $(n-j) \times (n-j)$, $j > 2$, minors for $n = 11$ fixed.

PROPOSITION 4.3. *If GE is applied on a CP binary Hadamard matrix of order 11, the third and fourth pivot from the end are 3 and $\frac{3}{2}$, respectively.*

*Proof.* Propositions 3.7 and 3.8, in combination with Lemma 4.1(ii), yield that for a CP binary Hadamard matrix $S$ of order $n = 11$ hold $S(n-3) = 2^{5-n}(n+1)^{\frac{n-5}{2}} = 27$

and $S(n-4) = 2^{8-n}(n+1)^{\frac{n-7}{2}} = 18$. By substituting these values, and also the general value for $S(n-2)$, in relation (4.1), we obtain

$$p_{n-2} = \frac{S(n-2)}{S(n-3)} = \frac{81}{27} = 3$$

and

$$p_{n-3} = \frac{S(n-3)}{S(n-4)} = \frac{27}{18} = \frac{3}{2}.$$

□

Next, we illustrate how the values of the $S(j)$ minors, $j = 1, \ldots, 6$, for a CP binary Hadamard matrix of order 11 can be specified, so that relation (4.1) can be used for the calculation of the first six pivots. First we give the following useful result.

LEMMA 4.4. *The maximum absolute value of the determinant of all $n \times n$ matrices with elements 0 and 1 is given in the following Table for $n = 1, \ldots, 6$.*

| n | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| max. det. | 1 | 1 | 2 | 3 | 5 | 9 |

More information and results about determinants of $(0, 1)$ matrices can be found in [1, 6, 11, 26].

PROPOSITION 4.5. *Let $S$ be a CP binary Hadamard matrix of order 11. Then $S(1) = 1$, $S(2) = 1$, $S(3) = 2$, $S(4) = 3$, $S(5) = 5$ and $S(6) = 9$.*

*Proof.* Consider as $S$ the third matrix of the Example in §1.1, which is a binary Hadamard matrix of order 11. We observe that $S(1) = 1$, $S(2) = 1$, $S(3) = 2$, $S(4) = 3$, $S(5) = 5$ and $S(6) = 9$. These values of the minors $S(j)$, $j = 1, \ldots, 6$, are the maximum values for $j \times j$, $j = 1, \ldots, 6$, matrices with elements 0 and 1, as it can be verified by Lemma 4.4.

We note that a binary Hadamard matrix of order 11 is unique up to equivalence since it is derived from a Hadamard matrix of order 12 being unique up to equivalence [25]. So the matrices with maximum determinants exist in every binary Hadamard matrix of order 11, since they have been proved to exist in one. If the matrix is CP, the matrices with maximum determinants must appear in the upper left corner, according to Lemma 4.1(ii), and this completes the proof. □

PROPOSITION 4.6. *Let $S$ be a CP binary Hadamard matrix of order 11. Then the first six pivots of $S$ are 1, 1, 2, $\frac{3}{2}$, $\frac{5}{3}$ and $\frac{9}{5}$.*

*Proof.* The pivot values in the enunciation are obtained by substituting appropriately the results of Proposition 4.5 in formula (4.1). □

PROPOSITION 4.7. *If GE with complete pivoting is performed on a binary Hadamard matrix of order 11 the pivot pattern is* $(1, 1, 2, \frac{3}{2}, \frac{5}{3}, \frac{9}{5}, 2, \frac{3}{2}, 3, 3, 6)$.

*Proof.* The first six pivots of a binary Hadamard matrix $S$ are given in Proposition 4.6 and the last four pivots in Proposition 4.3 and Theorem 4.2. The seventh pivot can be found from the property that the determinant of the matrix equals the product of the pivots, i.e.

$$\det S = \prod_{i=1}^{11} p_i \;\Rightarrow\; p_7 = \frac{\det S}{\prod_{i=1, i\neq 7}^{11} p_i} = 2.$$

□

Theorem 4.8. *If GE with complete pivoting is performed on a binary Hadamard matrix of order 11 (i.e. an SBIBD* $(11, 6, 3)$*) the growth factor is 6.*

*Proof.* Proposition 4.7 and the definition of the growth factor for CP matrices given in §4.1, taking into account that the element $a_{11}^{(0)}$ of a CP binary Hadamard matrix can be only 1, according to Lemma 4.1(ii), yield the requested growth factor. □

Following a similar, easier procedure, we can conclude the following pivot patterns as well.

Proposition 4.9. *If GE with complete pivoting is performed on binary Hadamard matrices of orders 3 and 7 (i.e. SBIBDs* $(3, 2, 1)$ *and* $(7, 4, 2)$*) the pivot patterns are* $(1, 1, 2)$ *and* $(1, 1, 2, 1, 2, 2, 4)$*, respectively.*

The importance of Theorem 4.8 and Proposition 4.9 is that they guarantee stability when solving linear systems with the respective matrices using GE with complete pivoting, since they have small growth factors (of order 1), and hence don't allow the existence of significant roundoff errors.

**5. Conclusions.** We proposed a technique for calculating all possible $(n - j) \times (n - j)$ minors of binary Hadamard matrices, which reveals also properties regarding their structure. The initial theoretical idea leads to an algorithm that overcomes the difficulties arising by the laborious calculations done by hand. The algorithm works theoretically for every pair of values $n$ and $j$. The usefulness of such a method is justified through the application to a problem of Numerical Linear Algebra, the growth problem. The results obtained in combination with the extensive numerical experiments lead to formulating the growth conjecture for binary Hadamard matrices.

An important open problem is whether there can be found a method to prove general results independent of the presence of parameters in the solution of the linear systems appearing in the method. Furthermore, it is a mystery why the value $(n+1)/4$ as fourth pivot from the end (cf. Tables 4.1 and 4.2) appears only for binary Hadamard matrices of order 15 obtained from Hadamard matrices of order 16 belonging to the I-class of equivalence, and specifically only in one pivot pattern. Finally, the approach of the conjecture for minors of Hadamard matrices, probably in connection with possible values of determinants of $\pm 1$ matrices, and the existence of binary Hadamard matrices for every $n \equiv 3 \pmod 4$ are open problems, too.

**Acknowledgments.** We thank the referees for the valuable comments and suggestions that contributed to a significant improvement of the presentation of this paper.

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